

# Strings in Yang-Mills-Higgs theory coupled to gravity

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Non-Abelian strings for an Einstein-Yang-Mills-Higgs theory are explicitly constructed. We consider  $N_f$  Higgs fields in the fundamental representation of the  $U(1) \times SU(N_c)$  gauge group in order to have a color-flavor  $SU(N_c)$  group remaining unbroken. Choosing a suitable ansatz for the metric, Bogomol'nyi-like first order equations are found and rotationally symmetric solutions are proposed. In the  $N_f = N_c$  case, solutions are local strings and are shown to be truly non-Abelian by parameterizing them in terms of orientational collective coordinates. When  $N_f > N_c$ , the solutions correspond to semilocal strings which, beside the orientational degrees of freedom, acquire additional collective coordinates parameterizing their transverse size. The low-energy effective theories for the correspondent moduli are found, showing that all zero modes are normalizable in presence of gravity, even in the semilocal case.

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## I. INTRODUCTION

It is well known that solitons (kinks, vortices, monopoles) play a central role in field theories, both at classical and quantum levels. In particular, a new type of (string-like) vortex solutions, called non-Abelian strings, was found quite recently in certain supersymmetric [1]-[4] and non-supersymmetric [5] gauge theories. More in detail, these strings arise in certain  $U(N_c)$  Yang-Mills-Higgs theories with  $N_f (\geq N_c)$  flavors and are mainly characterized by the presence of collective coordinates related to the orientation of the flux-tube in the internal color-flavor space. Due to these orientational moduli space, the above-mentioned strings behave as genuinely non-Abelian, leading to a number of new exciting phenomena: from confinement in  $\mathcal{N} = 1$  SQCD [6]-[7] and field-theoretic prototypes of D branes/strings [8],[9] to applications in cosmology as cosmic strings [10],[11]. Concerning the topic of cosmic strings (for a review and references see [12]-[14]), non-Abelian strings were originally introduced in this context as candidates to realize a mechanism proposed by Polchinski [14] through which gauge solitons could mimic the reconnection properties of fundamental strings.

The gravitational properties of vortex-like configurations were extensively studied in the past [12]. In a field-theoretic context, the simplest and most common model in which these configurations appear is the Einstein-Maxwell-Higgs model [15]-[21]. In the general case, Einstein-Maxwell-Higgs theories support two types of string solutions. These can be distinguished by their asymptotic geometries, which must be one of the two Levi-Civita metrics,

$$ds^2 = dt^2 - (dx^3)^2 - d\rho^2 - (a_1\rho + a_2)d\theta^2, \quad (1)$$

whence the cone, or

$$ds^2 = (b_1\rho + b_2)^{\frac{4}{3}}(dt^2 - (dx^3)^2) - d\rho^2 - (b_1\rho + b_2)^{-\frac{2}{3}}d\theta^2, \quad (2)$$

which is a Kasner metric. The Kasner branch does not have the required characteristics to describe a 'standard' cosmic string, and thus it is usually disregarded in physical applications. Each of the metrics (1) and (2) has two totally different behaviors. The behavior depends on the strength of the gravitational coupling, which is measured by the parameter  $G\xi$  (where  $\xi$  is the symmetry-breaking scale). That is, for  $G\xi \ll 1$ , which includes the GUT symmetry-breaking scale and most of the applications in cosmology,  $a_1$  and  $b_1$  are positives and then, (1) is the standard cone [15] and (2) is the asymptotic form of Melvin's magnetic universe [16]. For supermassive strings, which have  $G\xi \gtrsim 1$ ,  $a_1$  and  $b_1$  are negatives and then there is a conical singularity both in the Kasner-type metric [17] and in the conical metric [18]. When the Bogomol'nyi limit [19] of the Einstein-Maxwell-Higgs theory is considered, the critical coupling yields to a considerable simplification of the problem, since all second-order equations can be replaced by a curved-space analogue of the Bogomol'nyi equations [20]. String-like solutions in the Bogomol'nyi limit of the theory were analyzed in [20],[21] when only one Higgs field is present. The generalization to more than one Higgs field were considered in [22], where the solutions obtained correspond to gravitating semilocal vortices. An immediate consequence of the Bogomol'nyi equations is that the metric necessarily takes the conical asymptotic form (1) (see [20],[18]). This feature is also inherited by the supergravity extensions of the Maxwell-Higgs model, as was shown very recently in the embedding of Abelian cosmic strings into  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  supergravity [23].

On the other hand, analogous investigations on string solutions in Einstein-Yang-Mills-Higgs theories have remained, for some reason, rather incomplete (for a recent analysis of Einstein-Yang-Mills strings see [24]). It is the purpose of

this work to fill this gap looking for Einstein-Yang-Mills-Higgs theories which support local and semilocal non-Abelian string solutions. Since we are interested in the Bogomol'nyi limit of the model, only string metrics asymptotically conical will be analyzed. After presenting in section II the four dimensional Einstein-Yang-Mills-Higgs model with gauge group  $U(1) \times SU(N_c)$ ,  $N_f$  flavors and an arbitrary Higgs potential, we consider in section III a suitable ansatz for the metric which reduces the equations of motion to a set of first order (Bogomol'nyi) equations for a certain quartic potential. Next, in section IV we find gravitating local non-Abelian strings by considering a rotationally symmetric ansatz in the model with  $N_c = N_f$ . These strings are shown to have a non-Abelian character due to the existence of a set of orientational collective coordinates. In section V we proceed to do a similar analysis in the  $N_c < N_f$  case, yielding to gravitating semilocal non-Abelian strings. Apart from the orientational degrees of freedom, semilocal strings acquire new collective coordinates related to variations of the transverse size. In this case, we are able to find, in the large transverse size limit, explicit analytic solutions, not only for the matter fields, but also for the space-time metric. The section ends showing that in the limit of a very large transverse size of the string, semilocal solutions approximate two-dimensional sigma-model instantons on  $Gr(N_c, N_f)$  (i.e., Grassmannian lumps). In section VI we use the Manton procedure to obtain the low-energy effective theory of the moduli for both the local and semilocal cases. In the case of local strings, we have only orientational moduli and we find that the correspondent effective theory is a two dimensional  $\mathbf{CP}^{N_c-1}$  sigma-model, just the same as in flat space-time. For semilocal strings, where there are not only orientational modes but also size moduli, we find in the large transverse size limit three different effective theories depending on which the value of the parameter  $G\xi$  is. Quite remarkably, in contrast to what happens in flat space-time, all moduli become normalizable in the gravitating string case. Finally in section VII we present a summary and a discussion of our results.

## II. THE THEORY

The field content of the theory is given by a space-time metric  $g_{\mu\nu}$  where  $\mu, \nu, \dots = 0, 1, 2, 3$  are space-time indices, a  $SU(N_c) \times U(1)$  gauge field  $A_\mu$  and  $N_f$  complex scalars  $\phi$ . As well as the  $SU(N_c) \times U(1)$  gauge symmetry, the Lagrangian also enjoys a  $SU(N_f)$  flavor symmetry. Under these two groups, the scalar fields transform as  $(\mathbf{N}_c, \bar{\mathbf{N}}_f)$ . Thus,  $\phi$  can be seen as an  $N_c \times N_f$  matrix  $\phi = \phi_r^a$ , where the indices  $a, b, \dots = 1, 2, \dots, N_c$  refer to the gauge group and  $r, s, \dots = 1, 2, \dots, N_f$  to the flavor group.

We represent the gauge fields in terms of matrices in the fundamental representation of  $SU(N_c) \times U(1)$ , that is,  $A_\mu = A_\mu^A T^A + i/\sqrt{2N_c} A_\mu$ , where  $T^A$  ( $A, B, \dots = 1, 2, \dots, N_c^2 - 1$ ) are the generators of the  $\mathbf{N}_c$  representation of  $SU(N_c)$ . We use anti-Hermitian generators  $T^A$  satisfying

$$[(T^A)^a_b]^* = -(T^A)^b_a, \quad [T^A, T^B] = f^{ABC} T^C, \quad (T^A T^B)^a_a = -\frac{1}{2} \delta^{AB}. \quad (3)$$

The action takes the form

$$\mathcal{S} = \int d^4x \sqrt{g} \left\{ -\frac{1}{16\pi G} R - \frac{1}{4e_1^2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4e_2^2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^A F_{\rho\sigma}^A + \mathcal{D}_\mu \bar{\phi}^r \mathcal{D}^\mu \phi_r - V(\phi, \bar{\phi}) \right\} \quad (4)$$

where

$$\bar{\phi}_a^r \equiv (\phi_r^a)^*, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - f^{ABC} A_\mu^B A_\nu^C \quad (5)$$

and

$$\mathcal{D}_\mu \phi_r^a \equiv \partial_\mu \phi_r^a - \frac{i}{\sqrt{2N_c}} A_\mu \phi_r^a - A_\mu^A (T^A)^a_b \phi_r^b, \quad \mathcal{D}_\mu \bar{\phi}_a^r \equiv \partial_\mu \bar{\phi}_a^r + \frac{i}{\sqrt{2N_c}} A_\mu \bar{\phi}_a^r + A_\mu^A (T^A)^b_a \bar{\phi}_b^r. \quad (6)$$

Also, we use the conventions  $R_{\nu\rho\sigma}^\mu = \Gamma_{\nu\sigma,\rho}^\mu - \Gamma_{\nu\rho,\sigma}^\mu + \Gamma_{\omega\sigma}^\mu \Gamma_{\nu\rho}^\omega - \Gamma_{\omega\rho}^\mu \Gamma_{\nu\sigma}^\omega$ ,  $R_{\mu\nu} = R_{\mu\rho\nu}^\rho$ , signature  $(+, -, -, -)$  and  $g = -\det g_{\mu\nu}$ .

For simplicity we do not write gauge group indices which are summed, e.g.,

$$(\bar{\phi}^r T^A \phi_r) \equiv (T^A)^a_b \bar{\phi}_a^r \phi_r^b \quad (7)$$

The equations of motion obtained from the variation of the action are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu}^{\text{U}(1)} + T_{\mu\nu}^{\text{SU}(N_c)} + T_{\mu\nu}^{\text{mat}}), \quad (8)$$

$$\partial_\mu(\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}) = e_1^2\sqrt{g}j^\nu, \quad (9)$$

$$(\partial_\mu\delta^{AB} + f^{ABC}A_\mu^C)(\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}^B) = e_2^2\sqrt{g}j^{A\nu}, \quad (10)$$

$$[\mathcal{D}_\mu(\sqrt{g}g^{\mu\nu}\mathcal{D}_\mu\phi)]_r^a = -\sqrt{g}\frac{\partial V}{\partial\phi_r^a}, \quad (11)$$

where the stress-energy tensors and the gauge currents are given by

$$T_{\mu\nu}^{\text{U}(1)} = \frac{1}{e_1^2}\left(-F_{\mu\rho}F_\nu^\rho + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \quad (12)$$

$$T_{\mu\nu}^{\text{SU}(N_c)} = \frac{1}{e_2^2}\left(-F_{\mu\rho}^AF_\nu^{A\rho} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}^AF^{A\rho\sigma}\right), \quad (13)$$

$$T_{\mu\nu}^{\text{mat}} = \mathcal{D}_\mu\bar{\phi}^r\mathcal{D}_\nu\phi_r + \mathcal{D}_\nu\bar{\phi}^r\mathcal{D}_\mu\phi_r - g_{\mu\nu}\mathcal{D}_\rho\bar{\phi}^r\mathcal{D}^\rho\phi_r + g_{\mu\nu}V, \quad (14)$$

$$j^\mu = \frac{i}{\sqrt{2}N_c}(\mathcal{D}^\mu\bar{\phi}^r\phi_r - \bar{\phi}^r\mathcal{D}^\mu\phi_r), \quad (15)$$

$$j^{A\mu} = \mathcal{D}^\mu\bar{\phi}^rT^A\phi_r - \bar{\phi}^rT^A\mathcal{D}^\mu\phi_r. \quad (16)$$

In order to study cosmic string solutions, we assume the metric, gauge and matter fields to be static and symmetric under  $x^3$  translations. We also restrict to purely magnetic configurations. Thus, we will consider the following ansatz for the metric and gauge fields,

$$ds^2 = L^2dt^2 - K^2(dx^3)^2 + h_{ij}dx^idx^j, \quad (17)$$

$$A_\mu dx^\mu = A_id x^i, \quad i, j = 1, 2 \quad (18)$$

where the fields  $L$ ,  $K$ ,  $A_i$  and  $h_{ij}$  depend only on the two transverse coordinates  $x^k$  ( $k = 1, 2$ ).

With this ansatz the components of the Ricci tensor take the form

$$R_{00} = -L[(h^{ij}L_{,j})_{,i} + \gamma_{ik}^kh^{ij}L_{,j}] - \frac{L}{K}h^{ij}K_{,i}L_{,j} = -\frac{L}{K}(KL^{,i})_{,i}, \quad (19)$$

$$R_{33} = K[(h^{ij}K_{,j})_{,i} + \gamma_{ik}^kh^{ij}K_{,j}] + \frac{K}{L}h^{ij}L_{,i}K_{,j} = \frac{K}{L}(LK^{,i})_{,i}, \quad (20)$$

$$R_{ij} = r_{ij} - \frac{1}{L}(L_{,i,j} - \gamma_{ij}^kL_{,k}) - \frac{1}{K}(K_{,i,j} - \gamma_{ij}^kK_{,k}) \quad (21)$$

$$= r_{ij} - \frac{L_{,i;j}}{L} - \frac{K_{,i;j}}{K}, \quad (22)$$

$$R_{i0} = R_{i3} = 0, \quad R_{03} = 0, \quad (23)$$

where  $\gamma_{ij}^k$ ,  $r_{ij}$  and “;” denote the connection, Ricci tensor and covariant derivative corresponding to the two-dimensional transverse metric  $h_{ij}$ .

Concerning the gauge field strengths, their non-vanishing components are determined by a single magnetic component, that is,

$$F_{ij} = \epsilon_{ij}B, \quad (24)$$

$$F_{ij}^A = \epsilon_{ij}B^A, \quad (25)$$

where we have introduced the covariantly constant tensor field  $\epsilon_{ij} = -\epsilon_{ji}$ , normalized so that  $\epsilon_{ij}\epsilon^{jk} = \delta_i^k$ .

### III. BOGOMOL'NYI EQUATIONS

A significant simplification of this system is obtained if self-duality conditions are satisfied, i.e. if the system admits a Bogomol'nyi limit [19]. It is well known [20]-[21] that in the usual Abelian Higgs model, self-duality properties take

place in curved space-time if  $L(x^i)$  and  $K(x^i)$  are constants, say, 1. We will see now that the present generalized Higgs model has also a Bogomol'nyi limit if we keep

$$L(x^i) = 1, \quad K(x^i) = 1. \quad (26)$$

Using these conditions, it is easy to verify that  $R_{ij} = r_{ij}$  and  $R = r$ , these leading to that  $G_{ij}$  vanishes identically. Einstein equation (8) then implies the vanishing of  $T_{ij}$ , that is,

$$-\frac{1}{2e_1^2}B^2h_{ij} - \frac{1}{2e_2^2}B^2h_{ij} + \mathcal{D}_i\bar{\phi}^r\mathcal{D}_j\phi_r + \mathcal{D}_j\bar{\phi}^r\mathcal{D}_i\phi_r - h_{ij}\mathcal{D}_k\bar{\phi}^r\mathcal{D}^k\phi_r + h_{ij}V = 0 \quad (27)$$

The latter equation implies

$$\mathcal{D}_i\bar{\phi}^r\mathcal{D}_j\phi_r + \mathcal{D}_j\bar{\phi}^r\mathcal{D}_i\phi_r \propto h_{ij} \quad (28)$$

Now, without loss of generality, we may take the two-dimensional metric  $h_{ij}$  to be conformally flat, i.e.

$$h_{ij} = -\Omega^2\delta_{ij} \quad (29)$$

Then, we can write condition (28) as

$$\mathcal{D}_z\bar{\phi}^r\mathcal{D}_z\phi_r = 0 \quad (30)$$

with  $z = x + iy$ . The easiest way to solve this equation is by requiring either  $\mathcal{D}_z\phi_r^a = 0$  or  $\mathcal{D}_z\bar{\phi}_a^r = 0$ . Changing back to an arbitrary spatial coordinate system, we thus find the covariant self-duality condition for the Higgs field

$$\mathcal{D}_i\phi_r + i\eta\epsilon_i^j\mathcal{D}_j\phi_r = 0, \quad (31)$$

where  $\eta = \pm 1$  corresponds to self-dual or anti-self-dual solutions. Note however that, in contrast with what happens in the ( $N_f = 1$ ) Abelian case [25], equations (26) do not imply a priori the self-duality equations (31).

If we now return to eq.(27) and use the Higgs self-duality equation, we arrive to the following condition for the Higgs potential

$$V = \frac{B^2}{2e_1^2} + \frac{B^AB^A}{2e_2^2}. \quad (32)$$

In order to get a first order equation for the gauge fields, we consider the Higgs equations (11) and notice that for self-dual Higgs configuration they become

$$\left(B^A(T^A)_b^a + \frac{i}{\sqrt{2N_c}}B\delta_b^a\right)\phi_r^b = i\eta\frac{\partial V}{\partial\phi_r^a} \quad (33)$$

Now, thinking of  $B$  and  $B^A$  as functions of  $\phi$ , it follows from the Yang-Mills equations (9),(10) that they should be quadratic in  $\phi$ . In order to satisfy relation (33), we will consider a quartic Higgs potential for the theory. A generic quartic potential which respect both gauge and flavor invariances can be written as:

$$V(\phi, \bar{\phi}) = c_1 + c_2\bar{\phi}^r\phi_r + c_3(\bar{\phi}^r\phi_r)^2 + c_4(\bar{\phi}^rT^A\phi_r)^2 \quad (34)$$

From eq.(33) one obtains that

$$B = \eta\sqrt{2N_c}(c_2 + 2c_3\bar{\phi}^r\phi_r), \quad B^A = 2i\eta c_4\bar{\phi}^rT^A\phi_r. \quad (35)$$

Constants  $c_\alpha$  ( $\alpha = 1, \dots, 4$ ) are determined by requiring, firstly, that configurations satisfying self-dual conditions (31) and (35) be solutions of the Yang-Mills equations (9),(10) and secondly, that the Higgs potential has the minimum for  $\bar{\phi}^r\phi_r = N_c\xi$ . Then, self-duality equations for the gauge fields take the form

$$\begin{aligned} B &= \eta\frac{e_1^2}{\sqrt{2N_c}}(\bar{\phi}^r\phi_r - N_c\xi), \\ B^A &= -i\eta e_2^2\bar{\phi}^rT^A\phi_r, \end{aligned} \quad (36)$$

and the potential takes the form

$$V(\phi, \bar{\phi}) = \frac{e_1^2}{4N_c}(\bar{\phi}^r \phi_r - N_c \xi)^2 - \frac{e_2^2}{2}(\bar{\phi}^r T^A \phi_r)^2. \quad (37)$$

In summary, the self-duality first order equations for the Higgs and gauge fields are

$$\begin{aligned} \mathcal{D}_i \phi_r + i\eta \epsilon_i^j \mathcal{D}_j \phi_r &= 0, \\ B &= \frac{1}{2} \epsilon^{ji} F_{ij} = \eta \frac{e_1^2}{\sqrt{2N_c}}(\bar{\phi}^r \phi_r - N_c \xi), \\ B^A &= \frac{1}{2} \epsilon^{ji} F_{ij}^A = -i\eta e_2^2 \bar{\phi}^r T^A \phi_r, \end{aligned} \quad (38)$$

It is worth commenting that the existence of the first order equations (38) is strongly related to the possibility of having a locally supersymmetric theory whose bosonic sector coincides with our model. In fact, this supergravity theory could be used to obtain not only the Bogomol'nyi equations for the matter fields (38) but also Bogomol'nyi first-order equations for the gravitational field [28]. More precisely, one expects that the Bogomol'nyi equations could be obtained from the vanishing of the supersymmetric variation of the fermionic fields. In particular, the supersymmetric transformation of the gravitino should yields to a first-order Killing spinor equation for the supersymmetry parameter. Einstein equations will then be automatically satisfied as a consequence of the integrability condition of this Killing equation.

Clearly, the Higgs potential (37) is positive definite. Requiring its vanishing leads, due to the first term, to  $\phi$  to develop a vacuum expectation value (VEV) and, due to the second term, to the VEV to satisfy

$$\phi_r^a \bar{\phi}_b^r = \xi \delta_b^a. \quad (39)$$

Let us discuss shortly how the vacua of the Higgs potential (37) is and its dependence of  $N_c$  and  $N_f$ . It is clear from (39) that there is no vacua with vanishing potential for  $N_f < N_c$ , so this case is trivial. In the case of  $N_f = N_c$  there is an unique, isolated vacuum which, up to gauge transformations, takes the form

$$\phi_r^a = \sqrt{\xi} \delta_r^a. \quad (40)$$

The vacuum field (40) has the pattern of symmetry breaking [1]-[2]

$$U(1) \times SU(N_c) \times SU(N_f) \longrightarrow SU(N)_{c+f}, \quad (41)$$

where the surviving unbroken group  $SU(N)_{c+f}$  is a simultaneous gauge and flavor rotation. Due to this, the theory is said to lie in the colour-flavor locked phase. Finally, in the  $N_f > N_c$  case the theory has a Higgs branch of vacua, denoted  $\mathcal{N}_{N_c, N_f}$  [1]. For example, for Abelian theories, which support semi-local strings [26]-[27], the vacua is simply  $\mathcal{N}_{1, N_f} = \mathbf{CP}^{N_f-1}$ . In general, the Higgs branch is the Grassmannian of  $N_c$  planes in  $\mathbf{C}^{N_f}$ ,

$$\mathcal{N}_{N_c, N_f} = Gr(N_c, N_f) = \frac{SU(N_f)}{U(1) \times SU(N_c) \times SU(N_f - N_c)} \quad (42)$$

This is a symmetric space, and we may choose to work in any of the vacua without loss of generality. We pick,

$$\begin{aligned} \phi_r^a &= \sqrt{\xi} \delta_r^a & r &= 1, \dots, N_c \\ \phi_r^a &= 0 & r &= N_c + 1, \dots, N_f \end{aligned} \quad (43)$$

In this vacuum, the pattern of symmetry breaking is

$$U(1) \times SU(N_c) \times SU(N_f) \longrightarrow S[U(N_c)_{c+f} \times U(N_f - N_c)] \quad (44)$$

where  $S[\otimes_i U(N_i)]$  means we project out the diagonal, central  $U(1)$  from  $\otimes_i U(N_i)$ . Since the surviving unbroken group includes  $U(N_c)_{c+f}$ , in the  $N_f > N_c$  case the theory also lies in the colour-flavor locked phase.

Returning to the equations of motion, it is clear that with the Higgs potential eq.(37), condition (32) is automatically satisfied by self-dual gauge configurations. Thus, the only second-order equations that remain to solve are the 00 and 33 components of Einstein equation, which yields both to the following expression for the two-dimensional Ricci scalar  $r$ ,

$$r = 16\pi G(h^{ij} \mathcal{D}_i \bar{\phi}^r \mathcal{D}_j \phi_r - 2V) \quad (45)$$

Using the identity

$$\begin{aligned} h^{ij} \mathcal{D}_i \bar{\phi}^r \mathcal{D}_j \phi_r &= \frac{1}{2} h^{ij} (\mathcal{D}_i \bar{\phi}^r - i\eta \epsilon_i^l \mathcal{D}_l \bar{\phi}^r) (\mathcal{D}_j \phi_r + i\eta \epsilon_j^k \mathcal{D}_k \phi_r) \\ &\quad - i\eta B^A \bar{\phi}^r T^A \phi_r + \eta \frac{1}{\sqrt{2N_c}} B \bar{\phi}^r \phi_r + \eta \sqrt{\frac{N_c}{2}} \epsilon^{ik} j_{i;k} \end{aligned} \quad (46)$$

and the self-duality equations (38), we can rewrite the equation (45) for  $r$  as,

$$r = 8\pi \sqrt{2N_c} G \eta (\xi B + \epsilon^{ik} j_{i;k}), \quad (47)$$

where for self-dual configurations the  $U(1)$  current can be written as

$$j_i = -\frac{\eta}{\sqrt{2N_c}} \epsilon_i^j (\bar{\phi}^r \phi_r)_{,j}. \quad (48)$$

Now, in the conformal coordinate system  $r$  takes the simple form

$$r = \Omega^{-2} \Delta \log \Omega^2 \quad (49)$$

where  $\Delta$  is the flat-space Laplacian, i.e.,  $\Delta = \delta^{ij} \partial_i \partial_j$ . From eq.(47) we then get the following equation for  $\Omega^2$ :

$$\Delta(\log \Omega^2) = -8\pi G [\Delta(\bar{\phi}^r \phi_r) + \sqrt{2N_c} \eta \xi F_{12}] \quad (50)$$

### The Bogomol'nyi bound

As it is well known, the notion of energy in general relativity is more subtle than in special relativity. In the present case, since we are considering static axisymmetric matter configurations which tends asymptotically to their vacuum values, the two-dimensional transverse metric will tend asymptotically to that of a flat cone. Therefore, the deficit angle  $\delta$  may be taken as a measure of the gravitational energy per unit length (see [21] and reference therein). In our case, the deficit angle takes the form,

$$\begin{aligned} \frac{\delta}{8\pi G} &= \int d^2x \sqrt{g} T_0^0 \\ &= \int d^2x \sqrt{h} \left\{ \frac{1}{2e_1^2} B^2 + \frac{1}{2e_2^2} B^A B^A - h^{ij} \mathcal{D}_i \bar{\phi}^r \mathcal{D}_j \phi_r + V \right\} \end{aligned} \quad (51)$$

Now, by means of identity (46) and the Bogomol'nyi trick, the energy density  $T_0^0$  can be re-written as

$$\begin{aligned} T_0^0 &= \frac{1}{2e_1^2} [B - \eta \frac{e_1^2}{\sqrt{2N_c}} (\bar{\phi}^r \phi_r - N_c \xi)]^2 + \frac{1}{2e_2^2} [B^A + i\eta e_2^2 \bar{\phi}^r T^A \phi_r]^2 \\ &\quad + \frac{1}{2\sqrt{h}} |\mathcal{D}_i \phi_r + i\eta \epsilon_i^j \mathcal{D}_j \phi_r|^2 - \eta \sqrt{\frac{N_c}{2}} \xi B + \eta \sqrt{\frac{N_c}{2}} \epsilon^{ki} j_{i;k} \end{aligned} \quad (52)$$

Thus, integration of (52) yields a Bogomol'nyi bound for the energy, i.e.

$$\frac{\delta}{8\pi G} \geq \xi |\Phi|, \quad (53)$$

where  $\Phi$  is a topological number defined by

$$\Phi = \sqrt{\frac{N_c}{2}} \int d^2x \sqrt{h} B = \sqrt{\frac{N_c}{2}} \oint A_i d\sigma^i = 2\pi n. \quad (54)$$

As expected, we can see that the bound is saturated by configurations satisfying self-duality equations (38).

#### IV. NON-ABELIAN LOCAL STRINGS - $N_c = N_f = N$

In order to find non-Abelian vortex solutions to the self-duality equations, let us consider, following [2], rotationally symmetric configurations through the ansatz:

$$\begin{aligned}\phi &= \begin{pmatrix} \varphi(r) & 0 & \cdots & 0 & 0 \\ 0 & \varphi(r) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \varphi(r) & 0 \\ 0 & 0 & \cdots & 0 & e^{in\theta}\tilde{\varphi}(r) \end{pmatrix}, \\ A_i^A T^A &= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -(N-1) \end{pmatrix} \frac{i}{N} (\partial_i \theta) (f_N(r) - n), \\ A_i &= \sqrt{\frac{2}{N}} (\partial_i \theta) (-f(r) + n),\end{aligned}\tag{55}$$

where  $(r, \theta)$  are the polar coordinates in the two-dimensional transverse space.

Inserting this ansatz in the self-duality equations (38) we arrive to the first-order differential equations satisfied by the profile functions

$$\begin{aligned}r\partial_r\varphi + \frac{\eta}{N}(f - f_N)\varphi &= 0 \\ r\partial_r\tilde{\varphi} + \frac{\eta}{N}(f - (1-N)f_N)\tilde{\varphi} &= 0 \\ \frac{1}{r}\partial_r f + \eta\frac{e_1^2}{2}\Omega^2((N-1)\varphi^2 + \tilde{\varphi}^2 - N\xi) &= 0 \\ \frac{1}{r}\partial_r f_N + \eta\frac{e_2^2}{2}\Omega^2(\tilde{\varphi}^2 - \varphi^2) &= 0\end{aligned}\tag{56}$$

The boundary conditions at the origin follows from the requirement that the fields be nonsingular. This implies that

$$n\tilde{\varphi}(0) = 0, \quad f_N(0) = n, \quad f(0) = n.\tag{57}$$

At spatial infinity, configurations have to tend asymptotically to their vacuum values and then

$$\varphi(\infty) = \tilde{\varphi}(\infty) = \sqrt{\xi}, \quad f(\infty) = f_N(\infty) = 0\tag{58}$$

The first and second equation of (56) can be solved for the profiles of the gauge fields

$$\begin{aligned}f &= \frac{\eta}{2} r \partial_r ((1-N) \log \varphi^2 - \log \tilde{\varphi}^2), \\ f_N &= \frac{\eta}{2} r \partial_r (\log \varphi^2 - \log \tilde{\varphi}^2)\end{aligned}\tag{59}$$

We expect  $\tilde{\varphi}$  to have a zero only at  $r = 0$ , whereas  $\varphi$ , which does not wind, to have no zeros. Therefore, eq.(59) will be valid everywhere outside the origin.

Concerning field  $\Omega^2$ , after using eqs.(55) and (59) its equation of motion (50) becomes

$$\Delta\{\log \Omega^2 + 8\pi G[(N-1)\varphi^2 + \tilde{\varphi}^2 - \xi \log(\varphi^{2(N-1)}\tilde{\varphi}^2)]\} = 0\tag{60}$$

For a charge- $n$  vortex solution,  $\tilde{\varphi}$  will behave as  $\tilde{\varphi} \sim \text{const } r^{|n|}$  if  $r \rightarrow 0$ . It then follows that

$$\log \Omega^2 + 8\pi G[(N-1)\varphi^2 + \tilde{\varphi}^2 - \xi \log(\frac{\varphi^{2(N-1)}\tilde{\varphi}^2}{r^{2|n|}})]\tag{61}$$

is harmonic and bounded, hence is a constant. In particular this implies that the conformal factor has the following behavior at infinity

$$\Omega^2 \sim \text{const } r^{2(B-1)} \quad \text{if } r \rightarrow \infty,\tag{62}$$

where  $B = 1 - 8\pi|n|G\xi$ . Fixing the constant in eq.(62) to be  $\xi^{B-1}$ , we can write the asymptotic form of the metric as

$$ds^2 \sim dt^2 - (dx^3)^2 - (\xi r^2)^{B-1}(dr^2 + r^2 d\theta^2) \quad (63)$$

Thus, we find that the metric corresponding to a single non-Abelian local string has the same asymptotic behavior as that of the gravitating Abelian string (given by eq.(1)). It is characterized by the dimensionless parameter  $G\xi$ , which determines the strength of the gravitational coupling of the string [15]. If  $G\xi < \frac{1}{8\pi}$  (i.e.  $B > 0$ ), the metric is asymptotically conical. This can be easily seen by considering a new radial coordinate  $\rho$  given by  $\sqrt{\xi}\rho = B^{-1}(\sqrt{\xi}r)^B$ , which yields to the Minkowskian form (1) for the asymptotic metric:

$$ds^2 \sim dt^2 - (dx^3)^2 - d\rho^2 - B^2 \rho^2 d\theta^2. \quad (64)$$

The deficit angle corresponding to the metric (64) is

$$\delta = 2\pi(1 - B) = 16\pi^2|n|G\xi. \quad (65)$$

As the symmetry breaking scale grows,  $\delta$  exceeds  $2\pi$  and the conical picture of the string space-time must be abandoned. For a critical string (with  $G\xi = 1/8\pi$  and  $\delta = 2\pi$ ), the two-dimensional space is like a cylinder at infinite. Finally, over-critical strings have a deficit angle greater than  $2\pi$ , which happens for  $G\xi > 1/8\pi$ . This means that at infinity the space is like an inverted cone, closing up with a conical singularity which is at a finite proper distance. Those strings having a deficit angle  $\delta \geq 2\pi$  are known as supermassive strings [18]. Due to the presence of the singularity in the metric, supermassive strings appear to be of little physical interest.

Since the remaining Einstein equation can be integrated explicitly, we are thus left with a system of coupled equations for the Higgs profile functions,

$$\begin{aligned} \Delta \log(\varphi^{2(N-1)}\tilde{\varphi}^2) &= e_1^2 \Omega^2((N-1)\varphi^2 + \tilde{\varphi}^2 - N\xi) \\ \Delta \log(\varphi^{-2}\tilde{\varphi}^2) &= e_2^2 \Omega^2(\tilde{\varphi}^2 - \varphi^2), \end{aligned} \quad (66)$$

where  $\Omega^2$  is determined from eq.(61). Unfortunately, as in the flat case  $G = 0$  we have not been able to find analytical solutions. We can, however, establish from eq.(56) the asymptotic behavior of the solutions near  $r = 0$  and for large  $r$ . Near the polar axis, the first terms of the development in a power serie in  $r$  are

$$\begin{aligned} \varphi &\sim \varphi_0 + \frac{\varphi_0}{8N} ((e_2^2 - e_1^2)\varphi_0^2 + e_1^2 N(\varphi_0^2 - \xi)) r^2, \\ \tilde{\varphi} &\sim \tilde{\varphi}_0 r^{|n|}, \\ f &\sim n + \frac{\eta}{4} e_1^2 \Omega_0^2 (\varphi_0^2 - N(\varphi_0^2 - \xi)) r^2, \\ f_N &\sim n + \frac{\eta}{4} e_2^2 \Omega_0^2 \varphi_0^2 r^2, \quad r \rightarrow 0 \end{aligned} \quad (67)$$

where  $\varphi_0$  and  $\tilde{\varphi}_0$  are two arbitrary constants and  $\Omega_0^2 = \Omega^2(r = 0)$  could be determined through (61). Concerning the behavior for large  $r$ , the profile functions are modified by the non-trivial metric. Nevertheless, by rescaling the coordinates to be Minkowskian one can obtain the usual exponential behavior of the ANO strings [29]. Thus, using the radial coordinate  $\rho$  defined through  $\sqrt{\xi}\rho = B^{-1}(\sqrt{\xi}r)^B$  (for  $B > 0$ ), the behavior at large distances results

$$\begin{aligned} \varphi &\sim \sqrt{\xi} + \varphi_\infty \rho^{-\frac{1}{2}} (e^{-M_1 \rho} - e^{-M_2 \rho}), \\ \tilde{\varphi} &\sim \sqrt{\xi} + \varphi_\infty \rho^{-\frac{1}{2}} (e^{-M_1 \rho} - (1 - N)e^{-M_2 \rho}), \\ f &\sim \varphi_\infty \eta N e_1 B \rho^{\frac{1}{2}} e^{-M_1 \rho}, \\ f_N &\sim \varphi_\infty \eta N e_2 B \rho^{\frac{1}{2}} e^{-M_2 \rho}, \quad \rho \rightarrow \infty \end{aligned} \quad (68)$$

where  $\varphi_\infty$  is an arbitrary constant and  $M_i = e_i \sqrt{\xi}$  ( $i = 1, 2$ ). The dominant behavior of  $\varphi$ ,  $\tilde{\varphi}$  is given by the smallest exponential in (68)

Let us now discuss some facts about the vortex moduli space. While the vacuum is  $SU(N)_{\text{c+f}}$  symmetric, the solution given by eq.(55) breaks this symmetry down to  $U(1) \times SU(N-1)$ . This means that there exist a set of solutions with the same topological charge parameterized by the coset [1],[2]

$$\frac{SU(N)_{\text{c+f}}}{SU(N-1) \times U(1)} \cong \mathbf{CP}^{N-1} \quad (69)$$



Thus, if we suppose that the center-of-mass collective coordinates are decoupled, the moduli space in the case of a single unit charge vortex takes the form

$$\mathcal{M} \cong \mathbf{C} \times \mathbf{CP}^{N-1} \quad (70)$$

where  $\mathbf{C}$  parameterizes the center of mass of the vortex configuration. The presence of these extra orientational collective coordinates makes the vortices genuinely non-Abelian. One can make explicit the non-Abelian nature of the solution (55) by applying the color-flavor rotation preserving the asymmetric vacuum. To this end, it is convenient first to pass to the singular gauge where the scalar fields have no winding at infinite, while the vortex flux comes from the vicinity of the origin. Then, the Higgs and gauge fields can be written as

$$\begin{aligned} \phi &= \mathcal{U} \begin{pmatrix} \varphi(r) & 0 & \cdots & 0 & 0 \\ 0 & \varphi(r) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varphi(r) & \\ 0 & 0 & \cdots & 0 & \tilde{\varphi}(r) \end{pmatrix} \mathcal{U}^{-1}, \\ A_i^A T^A &= \mathcal{U} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -(N-1) \end{pmatrix} \mathcal{U}^{-1} \frac{i}{N} (\partial_i \theta) f_N(r), \\ A_i &= -\sqrt{\frac{2}{N}} (\partial_i \theta) f(r), \end{aligned} \quad (71)$$

where  $\mathcal{U} \in SU(N)$  parameterizes the orientational collective coordinates associated with the flux rotation in  $SU(N)$ . Following [5], we can parameterize these matrices as follows:

$$\frac{1}{N} \left\{ \mathcal{U} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -(N-1) \end{pmatrix} \mathcal{U}^{-1} \right\}^a_b = -n^a n_b^* + \frac{1}{N} \delta_b^a, \quad (72)$$

where  $n^a$  is a complex vector in the fundamental representation of  $SU(N)$ , and

$$n_a^* n^a = 1 \quad a = 1, \dots, N \quad (73)$$

Note that this gives the correct number of degrees of freedom for the charge-1 vortex case, namely,  $2(N-1)$ . With this parameterization the vortex solution (71) takes the form

$$\begin{aligned} \phi_b^a &= \frac{1}{N} (\tilde{\varphi}(r) + (N-1)\varphi(r)) \delta_b^a + (\tilde{\varphi}(r) - \varphi(r)) \left( n^a n_b^* - \frac{1}{N} \delta_b^a \right) \\ A_i^A (T^A)^a_b &= -i \left( n^a n_b^* - \frac{1}{N} \delta_b^a \right) \partial_i \theta f_N(r) \\ A_i &= -\sqrt{\frac{2}{N}} \partial_i \theta f(r) \end{aligned} \quad (74)$$

Note that the conformal factor  $\Omega^2$ , as obtained from eq.(61), results to be independent of the orientational collective coordinates  $n^a$ .

# V. NON-ABELIAN SEMILOCAL STRINGS - $N_c = N$ , $N_f = N + N_e$

We can easily write the extension of the ansatz (55) for the case  $N_f > N_c$  as follows

$$\begin{aligned} \phi &= \begin{pmatrix} \varphi(r) & 0 & \cdots & 0 & 0 & \rho_1^1(r) & \cdots & \cdots & \rho_{N_e}^1(r) \\ 0 & \varphi(r) & \cdots & 0 & 0 & \rho_1^2(r) & \cdots & \cdots & \rho_{N_e}^2(r) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \varphi(r) & 0 & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & e^{in\theta}\tilde{\varphi}(r) & \rho_1^N(r) & \cdots & \cdots & \rho_{N_e}^N(r) \end{pmatrix}, \\ A_i^A T^A &= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -(N-1) \end{pmatrix} \frac{i}{N} (\partial_i \theta) (f_N(r) - n), \\ A_i &= \sqrt{\frac{2}{N}} (\partial_i \theta) (-f(r) + n), \end{aligned} \quad (75)$$

With this ansatz, self-duality equations (31) become the following first-order equations for the profile functions

$$\begin{aligned} r\partial_r \varphi + \frac{\eta}{N} (f - f_N) \varphi &= 0 \\ r\partial_r \tilde{\varphi} + \frac{\eta}{N} (f - (1-N)f_N) \tilde{\varphi} &= 0 \\ r\partial_r \rho_r^a + \frac{\eta}{N} (f - f_N) \rho_r^a &= 0 \\ r\partial_r \rho_r^N + \frac{\eta}{N} (f - (1-N)f_N - Nn) \rho_r^N &= 0 \end{aligned} \quad (76)$$

where  $a = 1, \dots, N-1$  and  $r = 1, \dots, N_e$ . We also need to specify the boundary conditions which will determine the solutions in these equations. Is not difficult to see that in order to have nonsingular fields which tend asymptotically to vacuum configurations, the boundary condition for the Higgs profile functions are

$$\begin{aligned} n\tilde{\varphi}(0) &= 0, \quad \varphi(\infty) = \tilde{\varphi}(\infty) = \sqrt{\xi}, \\ \rho_r^a(\infty) &= \rho_r^N(\infty) = 0. \end{aligned} \quad (77)$$

Equations for  $\rho_r^a$  and  $\rho_r^N$  can be solve in terms of  $\varphi$  and  $\tilde{\varphi}$  through the relations

$$\rho_r^a(r) = \chi_r^a \varphi(r), \quad \rho_r^N(r) = \chi_r \frac{\tilde{\varphi}(r)}{r^{|n|}} \quad (78)$$

where  $\chi_r^a$  and  $\chi_r$  ( $a = 1, \dots, N-1$ ;  $r = 1, \dots, N_e$ ) are complex parameters. Now, the first relation in eq.(78) can only be compatible with boundary conditions (77) if  $\chi_r^a$ , and then  $\rho_r^a$ , are identically zero. Concerning gauge fields, the equations for their profile functions take now the form

$$\begin{aligned} \frac{1}{r} \partial_r f + \eta \frac{e_1^2}{2} \Omega^2 ((N-1)\varphi^2 + (1 + \frac{\bar{\chi}^r \chi_r}{r^{2|n|}}) \tilde{\varphi}^2 - N\xi) &= 0 \\ \frac{1}{r} \partial_r f_N + \eta \frac{e_2^2}{2} \Omega^2 ((1 + \frac{\bar{\chi}^r \chi_r}{r^{2|n|}}) \tilde{\varphi}^2 - \varphi^2) &= 0, \end{aligned} \quad (79)$$

which should be complemented with the boundary conditions

$$f_N(0) = f(0) = n, \quad f(\infty) = f_N(\infty) = 0 \quad (80)$$

Therefore, we get for the Higgs and gauge fields a family of solutions labelled by  $N_e$  complex parameters  $\chi_r$ , which, as we shall see, determine the size and orientation of the solutions. These string configurations are not conventional ANO strings, but, rather, semilocal strings (for a review of their properties and their relationship to electroweak strings, see [30]). These may be regarded as a hybrid of an ANO string and a sigma-model lump. As it is clear

from eqs.(75), (76) and (79), when the  $\chi_r$  parameters tend to zero, we reobtain the non-Abelian local string of the previous section. On the other hand, when  $|\chi_r|$  tends to infinite, solution (75) becomes a sigma-model lump on the target space  $\mathcal{N}_{N,N+N_e}$  (see below). Recall that, while the vortices are supported by  $\Pi_1(U(N)) = \mathbf{Z}$ , the lumps are supported by  $\Pi_2(\mathcal{N}_{N,N+N_e}) = \mathbf{Z}$ .

Concerning the space-time metric corresponding to these configurations, after getting the gauge profile functions  $f_N$  and  $f$  from the first and second equations of (76), the equation (50) for the conformal factor  $\Omega^2$  can be written as

$$\Delta\{\log \Omega^2 + 8\pi G[(N-1)\varphi^2 + (1 + \frac{\bar{\chi}^r \chi_r}{r^{2|n|}})\tilde{\varphi}^2 - \xi \log(\varphi^{2(N-1)}\tilde{\varphi}^2)]\} = 0 \quad (81)$$

Following the same reasoning as in the previous section we can infer that

$$\log \Omega^2 + 8\pi G[(N-1)\varphi^2 + (1 + \frac{\bar{\chi}^r \chi_r}{r^{2|n|}})\tilde{\varphi}^2 - \xi \log(\frac{\varphi^{2(N-1)}\tilde{\varphi}^2}{r^{2|n|}})] \quad (82)$$

is a constant. This leads to the same asymptotic behavior at infinity as that of the local strings, i.e.

$$\Omega^2 \sim \text{const } r^{2(B-1)} \quad \text{if } r \rightarrow \infty, \quad (83)$$

with  $B = 1 - 8\pi|n|G\xi$ . Therefore, the analysis of the asymptotic behavior of the string space as a function of  $G\xi$  done for local strings (see discussion after eq.(62)) is also valid to semilocal ones.

As in the case of the local strings, we can extract the asymptotic behavior of the gauge and Higgs fields from the first order eqs.(76) and (79). Near the polar axis, the behavior of the profile functions is

$$\begin{aligned} \varphi &\sim \varphi_0 + \frac{\varphi_0}{8N} ((e_2^2 - e_1^2)(\varphi_0^2 - \bar{\chi}^r \chi_r \tilde{\varphi}_0^2) + e_1^2 N(\varphi_0^2 - \xi)) r^2, \\ \tilde{\varphi} &\sim \tilde{\varphi}_0 r^{|n|}, \\ f &\sim n + \frac{\eta}{4} e_1^2 \Omega_0^2 (\varphi_0^2 - \bar{\chi}^r \chi_r \tilde{\varphi}_0^2 - N(\varphi_0^2 - \xi)) r^2, \\ f_N &\sim n + \frac{\eta}{4} e_2^2 \Omega_0^2 (\varphi_0^2 - \bar{\chi}^r \chi_r \tilde{\varphi}_0^2) r^2, \quad r \rightarrow 0 \end{aligned} \quad (84)$$

where  $\varphi_0$  and  $\tilde{\varphi}_0$  are arbitrary constants. In order to study the behavior at large distance, it is convenient to change coordinates in the same way as was done in the local string case. Thus, setting  $\sqrt{\xi}\rho = B^{-1}(\sqrt{\xi}r)^B$  (for  $B > 0$ ), the metric takes the asymptotic Minkowskian form (64) and the behavior of the profile functions at large  $\rho$  result

$$\begin{aligned} \varphi &\sim \sqrt{\xi} \left( 1 + \frac{2n^2(e_1^2 - e_2^2)}{N e_1^2 e_2^2} \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha-2} \right), \\ \tilde{\varphi} &\sim \sqrt{\xi} \left( 1 - \frac{1}{2} \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha} - \frac{2n^2((N-1)e_1^2 + e_2^2)}{N e_1^2 e_2^2} \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha-2} \right), \\ f &\sim n \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha} - \frac{2\eta(\alpha+2)Bn^2}{e_1^2} \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha-2}, \\ f_N &\sim n \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha} - \frac{2\eta(\alpha+2)Bn^2}{e_2^2} \xi^{|n|} \bar{\chi}^r \chi_r (B\sqrt{\xi}\rho)^{-\alpha-2}, \quad \rho \rightarrow \infty \end{aligned} \quad (85)$$

where  $\alpha = 2|n|B^{-1}$ . The resulting power-law decrease in the magnetic field at infinite is a significant departure from the usual exponential decay of the ANO string, which is associated with the confinement of magnetic flux [27]. Furthermore, semi-local strings develop additional collective coordinates  $\chi_r$  related to unlimited variations of their transverse size. The width of the flux tube results then completely undetermined, instead of being the Compton wave-length of the vector particle as in the ANO string case. This leads to a dramatic effect - semi-local strings, in contradistinction to the ANO strings, do not support linear confinement (see [31] for a nice analysis on deconfinement in the non Abelian semi-local string context).

In order to parameterize the semi-local string solution (75) in terms of the orientational collective coordinates, we apply a color-flavor  $SU(N)_{c+f}$  rotation preserving the vacuum (43). After going to the singular gauge and applying

the color-flavor rotation, the gauge and Higgs fields can be expressed as

$$\begin{aligned}
\phi_r^a &= (\tilde{\varphi}(r) - \varphi(r)) \left( n^a n_r^* - \frac{1}{N} \delta_r^a \right) \\
&\quad + \frac{1}{N} (\tilde{\varphi}(r) + (N-1)\varphi(r)) \delta_r^a \quad r = 1, \dots, N \\
\phi_r^a &= \tilde{\varphi}(r) \frac{e^{-in\theta}}{r^{|n|}} e^{i\delta} n^a \chi_r \quad r = N+1, \dots, N+N_e \\
A_i^A (T^A)^a_b &= -i \left( n^a n_b^* - \frac{1}{N} \delta_b^a \right) \partial_i \theta f_N(r) \\
A_i &= -\sqrt{\frac{2}{N}} \partial_i \theta f(r)
\end{aligned} \tag{86}$$

where we have used the same parametrization for the  $SU(N)_{c+f}$  matrices as in the previous section. Thus,  $n^a$  is a complex vector in the fundamental representation of  $SU(N)$  satisfying

$$n_a^* n^a = 1 \quad a = 1, \dots, N \tag{87}$$

Besides, the phase  $\delta$  is an arbitrary function of the orientational moduli, i.e.  $\delta = \delta(n, n^*)$ . This arbitrariness in the parametrization (72) will be useful when we study the low-energy effective action.

We can see that, in the case of charge-1 vortex configuration, eq.(86) gives the solution parameterized in terms of all the expected degrees of freedom, namely,  $2(N-1)$  orientational collective coordinates given by  $n^a$  and  $2N_e$  transverse size collective coordinates given by  $\chi_r$  (of course, we have not considered the two collective coordinates corresponding to the position of the center of mass).

### Grassmannian sigma-model lumps

As remarked before, in the limit of a very large transverse size of the string, solution (75) approximates a two-dimensional sigma-model instanton on the Higgs branch of vacua  $\mathcal{N}_{N, N+N_e} = Gr(N, N+N_e)$  lifted to four dimensions, i.e., a Grassmannian lump. Is the purpose of this section to get a deeper insight on this question. Indeed, we will be able, in the large transverse size limit, of solving in an explicit analytic form eqs.(76), (79) and (82) for the matter fields and the space-time metric, this yielding to a direct proof of the previous relation.

In order to do this, let us first assume the equality of the constant couplings,  $e_1 = e_2 = e$ . This greatly simplifies the problem without leading to a substantial loss of generality. After this assumption, it is convenient to define a new profile function  $k(r) = f(r) - f_N(r)$ , which in addition to  $\varphi$  satisfy the following equations

$$\begin{aligned}
r \partial_r \varphi + \frac{\eta}{N} k \varphi &= 0 \\
\frac{1}{r} \partial_r k + \eta \frac{e^2}{2} N \Omega^2 (\varphi^2 - \xi) &= 0
\end{aligned} \tag{88}$$

together with the boundary conditions

$$k(0) = 0, \quad k(\infty) = 0, \quad \varphi(\infty) = \sqrt{\xi}. \tag{89}$$

Clearly, the solutions for  $k$  and  $\varphi$  are those of the vacuum, that is,

$$k(r) \equiv 0, \quad \varphi(r) \equiv \sqrt{\xi}. \tag{90}$$

Concerning the rest of the equations, after using (90) they reduce to

$$\begin{aligned}
r \partial_r \tilde{\varphi} + \eta f \tilde{\varphi} &= 0 \\
\frac{1}{r} \partial_r f + \eta \frac{e^2}{2} \Omega^2 \left( \left( 1 + \frac{\bar{\chi}^r \chi_r}{r^2 |n|} \right) \tilde{\varphi}^2 - \xi \right) &= 0 \\
\log \Omega^2 + 8\pi G \left[ \left( 1 + \frac{\bar{\chi}^r \chi_r}{r^2 |n|} \right) \tilde{\varphi}^2 - \xi \log \left( \frac{\tilde{\varphi}^2}{r^2 |n|} \right) \right] &= \text{const}
\end{aligned} \tag{91}$$

with the boundary conditions

$$f(0) = n, \quad f(\infty) = 0, \quad \tilde{\varphi}(0) = 0, \quad \tilde{\varphi}(\infty) = \sqrt{\xi}. \quad (92)$$

Note that in the large lump limit, i.e.,  $\bar{\chi}^r \chi_r \gg (e^2 \xi)^{-|n|}$ , we can take

$$\frac{\bar{\chi}^r \chi_r r^{2(|n|-1)}}{e^2 \xi (r^{2|n|} + \bar{\chi}^r \chi_r)^2} \cong 0. \quad (93)$$

Then, the solutions of (91) has the form

$$\tilde{\varphi} = \sqrt{\xi} \frac{r^{|n|}}{\sqrt{r^{2|n|} + \bar{\chi}^r \chi_r}}, \quad (94)$$

$$f = \frac{n \bar{\chi}^r \chi_r}{r^{2|n|} + \bar{\chi}^r \chi_r}, \quad (95)$$

$$\Omega^2 = \text{const}(r^{2|n|} + \bar{\chi}^r \chi_r)^{-8\pi G \xi} \quad (96)$$

We then see that, as long as these expressions are valid, the Higgs field  $\phi$  in eq.(75) takes the form

$$\phi = \begin{pmatrix} \sqrt{\xi} & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{\xi} & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \sqrt{\xi} & 0 & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & e^{in\theta} \tilde{\varphi}(r) & \chi_1 \frac{\tilde{\varphi}(r)}{r^{|n|}} & \cdots & \cdots & \chi_{N_e} \frac{\tilde{\varphi}(r)}{r^{|n|}} \end{pmatrix}, \quad (97)$$

with  $\tilde{\varphi}$  given by eq.(94). Using this expression for  $\phi$  it is easy to verify that the vacuum condition

$$\phi_r^a \bar{\phi}_b^r = \xi \delta_b^a. \quad (98)$$

is satisfied at any  $r$ . Thus, the Higgs field define a map from the plane  $\mathbf{R}^2$  into the vacuum manifold  $Gr(N, N + N_e)$ . This map is analytic and of degree  $n$ , i.e. a charge- $n$  Grassmannian lump.

## VI. EFFECTIVE ACTION FOR LOW-ENERGY STRING DYNAMICS

Knowledge of the moduli space of the vortex configurations is a necessary ingredient in their applications in cosmology, as cosmic strings. Low-energy dynamics of vortices can be described by using the geodesic approximation due to Manton [32]. The main idea in [32] is to approximate the classical dynamics of solitons by their geodesic motion in the space of static/stationary solutions. In the case of vortex-like solutions, one method to do this is to assume that the collective coordinates are slow-varying functions of the string worldsheet coordinates  $t, x_3$ . Then, reinserting the ansatz in the original action, the moduli become fields of a (1+1)-dimensional sigma-model on the string world-sheet. This was the procedure followed in [2],[5] to study non-Abelian vortex dynamics in flat space-time. A generalization of the Manton's method to the case of gravitating solitons is, at the moment, not totally well developed (see recent works on a formal treatment in [33] and some previous applications to black-holes and  $\mathbf{CP}^1$  lumps in [34],[35]).

### World-sheet theory for local strings

In the case of local vortices ( $N_f = N_c$ ), we consider ansatz eq.(74) assuming that the orientational moduli are slowly varying functions of the string world-sheet coordinates, i.e.  $n^a = n^a(t, x^3)$ . Substituting the proposal (74) in the action (4) and performing the integral over the  $(x^1, x^2)$  plane, we end with a two-dimensional sigma-model for the  $n^a$  fields. Since  $n^a$  parameterize the string zero modes, no potential term is expected to be present in this sigma-model.

Now, since moduli parameters enter in (74) through a color-flavor rotation which now gets a dependence on the  $t$  and  $x^3$  coordinates, the original ansatz should be complemented by adding non-trivial  $A_0$  and  $A_3$  components to the gauge potential. Following [5] we propose

$$A_\alpha^A (T^A)^a_b = (\partial_\alpha n^a n_b^* - n^a \partial_\alpha n_b^* - 2n^a n_b^* (n^* \partial_\alpha n)) \rho(r) \quad \alpha = 0, 3 \quad (99)$$

where a new profile function  $\rho(r)$  has been introduced, to be determined by its equation of motion through a minimization procedure.

From the  $SU(N)$  gauge field strength

$$F_{\alpha i}^A(T^A)_b^a = -(\partial_\alpha n^a n_b^* - n^a \partial_\alpha n_b^* - 2n^a n_b^* (n^* \partial_\alpha n)) \partial_i \rho(r) - i(\partial_\alpha n^a n_b^* + n^a \partial_\alpha n_b^*) \partial_i \theta f_N (1 - \rho) \quad (100)$$

we see that  $\rho(r)$  has to satisfy

$$\rho(0) = 1 \quad \rho(\infty) = 0 \quad (101)$$

in order to have a finite contribution in the action.

After inserting the modified ansatz in the action (4), we get the low-energy effective action

$$\mathcal{S}_{eff} = 2\beta \int dt dx^3 (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^*)) \quad (102)$$

The constant coupling  $\beta$  is related to the four-dimensional coupling  $e_2^2$  through the relation

$$\beta = \frac{2\pi}{e_2^2} I \quad (103)$$

where  $I$  is the integral

$$I = \int_0^\infty r dr [(\partial_r \rho)^2 + \frac{1}{r^2} f_N^2 (1 - \rho)^2 + \frac{e_2^2}{2} \Omega^2 (2(\tilde{\varphi} - \varphi)^2 (1 - \rho) + (\tilde{\varphi}^2 + \varphi^2) \rho^2)] \quad (104)$$

The integral  $I$  can be viewed as an action for the profile function  $\rho$ . Thus,  $I$  is extremized by configurations  $\rho$  satisfying the second-order equation

$$-\frac{d^2 \rho}{dr^2} - \frac{1}{r} \frac{d\rho}{dr} - \frac{1}{r^2} f_N^2 (1 - \rho) + \frac{e_2^2}{2} \Omega^2 ((\tilde{\varphi}^2 + \varphi^2) \rho - (\tilde{\varphi} - \varphi)^2) = 0 \quad (105)$$

As done in [3] in flat space-time, using the first-order equations (56) one can show that the solution of (115) is given by

$$\rho = 1 - \frac{\tilde{\varphi}}{\varphi} \quad (106)$$

Besides, this solution satisfies the boundary conditions (101). Substituting this solution back into the expression (104) for the integral  $I$ , one can check that this integral reduces to a total derivative and is given by the flux of the string. That is,

$$I = \int_0^\infty dr \left[ 2\partial_r \left( \frac{\tilde{\varphi}}{\varphi} \right) \left( -\eta \frac{\tilde{\varphi}}{\varphi} f_N \right) + \left( 1 - \left( \frac{\tilde{\varphi}}{\varphi} \right)^2 \right) \eta \partial_r f_N \right] = -\eta \left[ \left( \frac{\tilde{\varphi}}{\varphi} \right)^2 f_N - f_N \right] \Big|_0^\infty = |n| \quad (107)$$

Recalling the world-sheet effective theory for the orientational coordinates given by eq.(102), it corresponds to the two dimensional  $\mathbf{CP}^{N-1}$  sigma-model, as was already anticipated by using symmetry arguments in section IV. This can be easily seen from the invariance of the action (102) under the  $U(1)$  gauge transformations

$$n^a \rightarrow e^{i\vartheta(t, x^3)} n^a, \quad n_a^* \rightarrow e^{-i\vartheta(t, x^3)} n_a^* \quad (108)$$

and the constraint  $n^a n_a^* = 1$  for the fields. As shown in [2], the  $\mathbf{CP}^{N-1}$  sigma-model is also the theory governing the effective vortex dynamics in flat space-time. As a result, at this level of approximation, the dynamics of the orientational moduli of a single local string does not seem to be affected by the presence of the gravitational coupling. Several properties of the dynamics of the theory depend on how the moduli space is (like, for instance, the probability of reconnection of cosmic strings in the low-energy regimen). Thus, an unchanged moduli space in the case of an arbitrary number of solitons would imply that there is no variation in the low-energy physics of gravitating local strings with respect to that of local strings in flat space-time. We shall see below that this situation changes considerably in the case of semilocal vortices.

### World-sheet theory for semilocal strings

Let us study now the case  $N_f > N_c$ . In this case, we have to consider both the orientational  $n^a$  and the size moduli  $\chi_r$  as slowly varying functions of the string world-sheet coordinates  $t, x^3$ . For simplicity we will take the gauge constant couplings to be equal,  $e_1 = e_2 = e$ . Then, we can put  $f_N = f$  and  $\varphi = \sqrt{\xi}$  in the expressions of eq.(86) for the fields in the singular gauge. Besides, an ansatz for the components  $A_\alpha$  ( $\alpha = 0, 3$ ) of the gauge field must be proposed, so we will consider the same expression (99) as in the local string case. Concerning the space-time metric, due to the dependence of the conformal factor  $\Omega^2$  with the size moduli  $\chi_r$  (see eq.(82)), in the case of semilocal strings also the metric become world-sheet coordinate dependent.

Thus, introducing the ansatz (86) and (99) in the action (4) we arrive to the effective action for the moduli coordinates. The corresponding Lagrangian  $\mathcal{L}_{eff}$  can be decomposed in two parts,

$$\mathcal{L}_{eff} = \mathcal{L}_{\chi,n} + \mathcal{L}_\chi, \quad (109)$$

the first part  $\mathcal{L}_{\chi,n}$  given the effective action for the orientational coordinates  $n^a$  and the second one  $\mathcal{L}_\chi$  including the kinetic terms of the size moduli  $\chi_r$  and being independent of  $n^a$ . The expressions for these Lagrangians are

$$\begin{aligned} \mathcal{L}_{\chi,n} = & 2\beta I_0 (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^*)) \\ & + \beta I_1 (n^* \partial^\alpha n + i \partial^\alpha \delta) (\partial_\alpha \bar{\chi}^r \chi_r - \bar{\chi}^r \partial_\alpha \chi_r + \bar{\chi}^r \chi_r (n^* \partial_\alpha n + i \partial_\alpha \delta)) \end{aligned} \quad (110)$$

$$\begin{aligned} \mathcal{L}_\chi = & -\frac{1}{8G} \int_0^\infty r dr \left( \frac{1}{2} \Omega^{-2} \partial^\alpha (\Omega^2) \partial_\alpha (\Omega^2) - 2 \partial^\alpha \partial_\alpha (\Omega^2) + \frac{1}{r} \partial_r (r \partial_r (\log \Omega^2)) \right) \\ & + \beta \int_0^\infty r dr \left\{ \frac{1}{r^2} \partial^\alpha f \partial_\alpha f + e^2 \Omega^2 \left[ \partial^\alpha \tilde{\varphi} \partial_\alpha \tilde{\varphi} + \frac{1}{r^{2|n|}} \partial^\alpha (\tilde{\varphi} \bar{\chi}^r) \partial_\alpha (\tilde{\varphi} \chi_r) \right] \right. \\ & \left. - e^2 \left[ \partial_r \tilde{\varphi} \partial_r \tilde{\varphi} + \left( \frac{1}{r} f \tilde{\varphi} \right)^2 + (\partial_r \tilde{\varphi} - |n| \frac{1}{r} \tilde{\varphi})^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} + \left( \frac{1}{r} (f - n) \tilde{\varphi} \right)^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \right] \right\} \end{aligned} \quad (111)$$

where  $\beta = 2\pi/e^2$  and  $I_i$  ( $i = 0, 1$ ) are the integrals given by

$$I_0 = \int_0^\infty r dr [(\partial_r \rho)^2 + \frac{1}{r^2} f^2 (1 - \rho)^2 + \frac{e^2}{2} \Omega^2 (2(\tilde{\varphi} - \sqrt{\xi})^2 (1 - \rho) + (\tilde{\varphi}^2 + \xi) \rho^2 + \tilde{\varphi}^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} (1 - \rho)^2)] \quad (112)$$

$$I_1 = e^2 \int_0^\infty r dr \frac{\Omega^2}{r^{2|n|}} \tilde{\varphi}^2 \quad (113)$$

The first term in  $\mathcal{L}_{\chi,n}$  coincides with the Lagrangian of a two dimensional  $\mathbf{CP}^{N-1}$  sigma-model for the fields  $n^a$  multiplied by the integral  $I_0$  depending on the size moduli  $\chi_r$ . On the other hand, the second term in  $\mathcal{L}_{\chi,n}$  includes mixed kinetic terms between the orientational moduli  $n^a$  and the semilocal size  $\chi_r$ , which are similar to those found in [36]. In our case, the second term in  $\mathcal{L}_{\chi,n}$  can be eliminated by choosing the phase  $\delta$  such that

$$\partial_\alpha \delta = i n^* \partial_\alpha n. \quad (114)$$

Concerning  $\mathcal{L}_\chi$ , the first line in (111) represents the contribution of the Hilbert-Einstein term in (4), while the second and third lines come from the Yang-Mills-Higgs sector. It is clear that from the first line in  $\mathcal{L}_\chi$  only the first term has to be considered, since the rest are total derivatives. Besides, one can put the conformal factor  $\Omega^2$  in terms of  $\tilde{\varphi}$  and  $\chi_r$  just by using the relation (82).

As we did in the  $N_f = N_c$  case, we consider the integral  $I_0$  as an action for the profile function  $\rho$ . Thus, from  $I_0$  one obtains the second-order equation which the function  $\rho$  must satisfy, namely,

$$-\frac{d^2 \rho}{dr^2} - \frac{1}{r} \frac{d\rho}{dr} - \frac{1}{r^2} f^2 (1 - \rho) + \frac{e^2}{2} \Omega^2 ((\tilde{\varphi}^2 + \xi) \rho - (\tilde{\varphi} - \sqrt{\xi})^2 - \tilde{\varphi}^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} (1 - \rho)) = 0 \quad (115)$$

with the boundary conditions  $\rho(0) = 1$ ,  $\rho(\infty) = 0$ . This equation is solved by

$$\rho = 1 - \frac{\tilde{\varphi}}{\sqrt{\xi}} \quad (116)$$

as one can show using the first-order equations (76) and (79) for the profile functions. Substituting this solution back into  $I_0$ , one finds that this integral reduce to

$$\begin{aligned} I_0 &= \int_0^\infty dr \left[ 2\partial_r \left( \frac{\tilde{\varphi}}{\sqrt{\xi}} \right) \left( -\eta \frac{\tilde{\varphi}}{\sqrt{\xi}} f \right) + \left( 1 - \frac{\tilde{\varphi}^2}{\xi} \right) \eta \partial_r f + r \frac{e^2}{2} \Omega^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \tilde{\varphi}^2 \right] \\ &= |n| + \frac{e^2}{2} \int_0^\infty r dr \Omega^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \tilde{\varphi}^2 \end{aligned} \quad (117)$$

Using the latter expression for  $I_0$  and the phase  $\delta$  given by eq.(114), the Lagrangian  $\mathcal{L}_{\chi,n}$  takes now the simpler form

$$\mathcal{L}_{\chi,n} = 2\beta(|n| + \frac{e^2}{2} \int_0^\infty r dr \Omega^2 \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \tilde{\varphi}^2) (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^* n)) \quad (118)$$

At this point we are ready to face an important question about semilocal strings, which is the normalizability of the zero modes. Recently, the moduli space of semilocal non-Abelian strings in flat space-time ( $\Omega^2 = 1$ ) was obtained through the Manton procedure in [11],[31],[36]. Although some disagreements between the results of these works, all of them found that some of the orientational and size zero modes are non-normalizable. In fact, they found that a single semilocal vortex always has only non-normalizable moduli. The non-normalizability of a zero mode manifests through an infinite kinetic term for this mode, due to logarithmic divergences in the infrared. It follows then that the corresponding moduli become frozen in this approximation, since any change in it is impeded by infinite inertia. On the other hand, it was noted in the case of self-gravitating  $\mathbf{CP}^1$  lumps [35] that the deformation of the space-time introduced by gravity is just sufficient to remove the singularity. One expects, therefore, that also in the case of semilocal strings the previously frozen moduli defrost, once gravitational effects are taken into account [39]. In fact, one can see that this is what actually happens by using the asymptotic expansions in the infrared obtained from eqs.(83) and (85),

$$\begin{aligned} \tilde{\varphi} &\sim \sqrt{\xi} - \frac{\sqrt{\xi}}{2} \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \\ f &\sim n \frac{\bar{\chi}^r \chi_r}{r^{2|n|}} \\ \Omega^2 &\sim \text{const } r^{2(B-1)} \quad \text{if } r \rightarrow \infty \end{aligned} \quad (119)$$

where  $B = 1 - 8\pi|n|G\xi$  (note that these expansions are valid if  $B > 0$ ). Thus, introducing these expansions in the expressions (111) and (118) for the effective Lagrangians, it is easily verified that all modes become normalizable when strings are coupled to gravity.

Coming back to  $\mathcal{L}_\chi$ , in order to write this Lagrangian as a sigma-model one, it would be necessary to put expression (111) explicitly in terms of the field  $\chi_r$ . Unfortunately, this is not possible since it is not known how the profile functions  $f$  and  $\tilde{\varphi}$  depend on the size moduli. We can, however, make use of the large transverse size limit

$$\bar{\chi}^r \chi_r m_W^2 \gg 1 \quad (120)$$

(where we have taken the winding  $n$  to be 1 and called  $m_W^2 = e^2 \xi$ ). As was shown previously, in this limit we have explicit analytic solutions given by eqs.(94)-(96), not only for the matter fields, but also for the space-time metric. Concerning this last field, we choose the arbitrary constant in (96) so that the conformal factor takes the form

$$\Omega^2 = \frac{1}{(m_W^2(r^2 + \bar{\chi}^r \chi_r))^{8\pi G\xi}} \quad (121)$$

It is convenient to recall that the behavior of the corresponding two-dimensional metric  $h_{ij} = \Omega^2 \delta_{ij}$  depends on the value of the parameter  $G\xi$  (or, equivalently, on the value of the deficit angle  $\delta$ ). If  $G\xi < 1/8\pi$ , this metric is asymptotically conical, with a deficit angle  $\delta = 16\pi^2 G\xi$ . If  $G\xi = 1/8\pi$ , the deficit angle is  $2\pi$  and then the two-dimensional transverse space is asymptotically cylindrical. Finally, a deficit angle greater than  $2\pi$ , which happens for  $G\xi > 1/8\pi$ , means that at infinity the space is like an inverted cone, with a conical singularity at a finite proper distance.

Thus, introducing the expressions (94)-(96) for the profile functions in (111) and (118), we obtain the following effective Lagrangians  $\mathcal{L}_\chi$  and  $\mathcal{L}_{\chi,n}$ :

$$\begin{aligned} \beta^{-1} \mathcal{L}_\chi &= \left( \frac{1}{12} - \frac{(9-B)(m_W^2 \bar{\chi}^r \chi_r)^B}{8(2-B)} \right) \frac{\partial^\alpha (\bar{\chi}^s \chi_s) \partial_\alpha (\bar{\chi}^t \chi_t)}{(\bar{\chi}^u \chi_u)^2} + \frac{(m_W^2 \bar{\chi}^r \chi_r)^B}{1-B} \frac{\partial^\alpha \bar{\chi}^s \partial_\alpha \chi_s}{\bar{\chi}^t \chi_t} - m_W^2 \\ \beta^{-1} \mathcal{L}_{\chi,n} &= 2 \left( 1 + \frac{(m_W^2 \bar{\chi}^r \chi_r)^B}{4(1-B)} \right) (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^* n)) \end{aligned} \quad (122)$$



Finally, depending again on the value of the parameter  $G\xi$ , three different theories result from (122) for the low-energy effective dynamics of the vortex in the large transverse size limit:

- First case:  $0 < B < 1$  or  $G\xi < \frac{1}{8\pi}$ , metric asymptotically conical

$$\beta^{-1}\mathcal{L}_{eff} = \frac{(m_W^2 \bar{\chi}^r \chi_r)^B}{1-B} \left( \frac{\partial^\alpha \bar{\chi}^s \partial_\alpha \chi_s}{\bar{\chi}^t \chi_t} - \frac{(1-B)(9-B)}{8(2-B)} \frac{\partial^\alpha (\bar{\chi}^s \chi_s) \partial_\alpha (\bar{\chi}^t \chi_t)}{(\bar{\chi}^u \chi_u)^2} + \frac{1}{2} (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^* n)) \right) \quad (123)$$

- Second case:  $B = 0$  or  $G\xi = \frac{1}{8\pi}$ , metric asymptotically cylindrical

$$\beta^{-1}\mathcal{L}_{eff} = \frac{\partial^\alpha \bar{\chi}^r \partial_\alpha \chi_r}{\bar{\chi}^s \chi_s} - \frac{23}{48} \frac{\partial^\alpha (\bar{\chi}^r \chi_r) \partial_\alpha (\bar{\chi}^s \chi_s)}{(\bar{\chi}^t \chi_t)^2} + \frac{5}{2} (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^* n)) \quad (124)$$

- Third case:  $B < 0$  or  $G\xi > \frac{1}{8\pi}$ , metric asymptotically conically singular

$$\beta^{-1}\mathcal{L}_{eff} = \frac{1}{12} \frac{\partial^\alpha (\bar{\chi}^r \chi_r) \partial_\alpha (\bar{\chi}^s \chi_s)}{(\bar{\chi}^t \chi_t)^2} + 2 (\partial^\alpha n^* \partial_\alpha n - (n^* \partial^\alpha n)(\partial_\alpha n^* n)) \quad (125)$$

It is interesting to note that for supermassive strings ( $G\xi \geq 1/8\pi$ ) the dynamics of the orientational modes decouples from that of the size moduli. Moreover, the Lagrangian for the orientational moduli of supermassive semilocal strings corresponds to that of the two-dimensional  $\mathbf{CP}^{N-1}$  sigma-model, as in the case of local strings.

On the other hand, the most relevant case is the first one, with  $G\xi < 1/8\pi$ , since it includes the range of physical applications (e.g. cosmological data gives an upper limit  $G\xi < 10^{-6}$  for cosmic strings -see [14] and references therein). In this case, Lagrangian (123) presents mixed terms between size and orientational moduli turning out the theory far more complex. Indeed, one can expect this kind of terms in the Lagrangian since our effective theory must be considered as a deformation of the theory obtained in flat space-time in [31],[36].

From expressions (123)-(125) it is clear that in the large transverse size limit all modes are normalizable, for any value of the parameter  $B$ . This could have several consequences in the low-energy physics of the theory. If a similar defrosting effect takes place also in the moduli space of more than one string coupled to gravity, several previous analysis of the dynamics of strings (like those done in [10],[11] where reconnection of non-Abelian cosmic strings were studied) could be considerably affected. Thus, we see that, in contrast to what happens for a single local string, the presence of gravity produce important changes in the moduli space of semilocal strings, which can be relevant to the physical properties of this kind of topological defect.

## VII. SUMMARY AND DISCUSSION

The main goal of this work was the construction of a new type of gravitating string solutions, mainly characterized by being genuinely non-Abelian. To do this, we considered a four dimensional Einstein-Yang-Mills-Higgs theory with gauge group  $G = U(1) \times SU(N_c)$ ,  $N_f$  scalar fields in the fundamental representation of  $G$  and an *a priori* undetermined Higgs potential.

Guided by results obtained in the Abelian case [20],[21], we proposed an ansatz for the space-time metric which allows us to find first-order Bogomol'nyi equations from consistency conditions resulting from the (highly complex) second order Euler-Lagrange equation of motion.

Not quite surprisingly, consistency fixes the Higgs potential to be the quartic one given by eq.(37). In particular, the resulting potential yields to a pattern of symmetry breaking containing, in the  $N_f \geq N_c$  case, a surviving unbroken group  $SU(N)_{c+f}$ . This property of the Higgs potential is completely necessary in order to find strings solutions having an internal orientational moduli space. Moreover, as it is shown in section III the gravitational energy (associated to the total deficit angle) has, for such a potential, a lower bound which is a topological number related to the magnetic flux. *A posteriori*, we verified that the Bogomol'nyi bound is saturated precisely by configurations satisfying the first order equations.

In order to solve the Bogomol'nyi equations, we proposed a rotationally symmetric ansatz similar to that used in flat space-time to get non-Abelian vortices [2]. In the  $N_f = N_c$  case, we showed that this ansatz yields to gravitating local non-Abelian strings. The non-Abelian character becomes apparent from the existence of a set of orientational collective coordinates parameterizing the string solution. Concerning the large distance behavior, these strings are similar to ANO strings, in the sense that they have the usual exponential decay.

With respect to the  $N_c < N_f$  case, a generalization of the ansatz allowed us to construct gravitating semilocal non-Abelian strings. In this case, semilocal strings acquire, beside the orientational degrees of freedom, new collective

coordinates related to variations of the transverse size. It is worth noting that this type of strings has a decreasing power-law large-distances behavior, this resulting in the deconfinement of the magnetic flux. In fact, the width of the magnetic flux results completely undetermined since the new collective coordinates permit to do unlimited variations of the transverse size of the cosmic string. We were able to find in the large transverse size limit explicit analytic solutions, not only for the matter fields, but also for the space-time metric. Interestingly enough, the explicit solutions allowed us to clearly show that semilocal solutions approximate two-dimensional sigma-model instantons on the Higgs branch of vacua  $Gr(N_c, N_f)$  in the limit of a large transverse size of the string.

Finally, string world-sheet effective actions for the moduli coordinates were obtained using the Manton procedure. In the case of local strings, the dynamics of the orientational moduli turned out to be that of a two-dimensional  $\mathbf{CP}^{N_c-1}$  sigma-model, which is just the same effective theory governing the dynamics of strings in the flat space-time case.

In contrast, when semilocal strings are considered, gravitational effects already arise at a low-energy level, radically changing the moduli dynamics. Indeed, in flat space-time low energy dynamics of orientational and size moduli is highly constrained due to non-normalizability of some (and sometimes all) of the zero-modes. We found that gravity alters completely this situation since, quite surprisingly, all orientational and size modes previously frozen become normalizable when strings are coupled to gravity.

In view of the results described above, it would be interesting to study their relevance concerning the physics of cosmic strings. For instance, gravity could induce changes in the moduli space leading to a probability of reconnection  $P < 1$ . To analyze this, it would be necessary to look for solutions corresponding to composite gravitating non-Abelian vortices, analogous to those found in flat space-time in [37], in order to obtain the dynamics of two intersecting vortices. In this respect, it could be quite useful to search also for a generalization on the moduli matrix approach [38] to the case of gravitating solitons. We hope to report on this issues in a forthcoming work.

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